Hamiltonian embedding of the massive noncommutative U(1) theory

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Abstract

We show that the massive noncommutative U(1) can be embedded in a gauge theory by using the BFFT Hamiltonian formalism. By virtue of the peculiar non-Abelian algebraic structure of the noncommutative massive $U(1)$ theory, several specific identities involving Moyal commutators had to be used in order to make the embedding possible. This leads to an infinite number of steps in the iterative process of obtaining first-class constraints. We also shown that the involutive Hamiltonian can be constructed.

I. INTRODUCTION

The embedding procedure is an interesting mechanism where theories with second-class constraints [1] are transformed into more general (gauge) theories in such a way that all constraints become first-class. Once this is achieved it is possible to exploit the machinery available for quantizing first-class theories [2]. The use of Dirac brackets is completely avoided. The important point of this mechanism is that the general theory so obtained contains all the physics of the embedded one.

A systematic formalism for embedding was developed by Batalin, Fradkin, Fradkina, and Tyutin (BFFT) [3,4]. A fundamental characteristic of the method is the extension of the original phase space by introducing one auxiliary field for each second-class constraint. The BFFT method is quite elegant and the obtainment of first-class constraints is done in an iterative way. The first correction is linear in the auxiliary variables, the second one is quadratic, and so on. In the case of systems with just linear constraints, like chiral-bosons [5], only linear corrections are enough to make them first-class [6,7]. Here, we mention that the method is equivalent to express the dynamic quantities by means of shifted coordinates [8].

However, for systems with nonlinear constraints, the iterative process may go beyond the first correction. This is a crucial point because the first iterative step does not give a unique solution and we do not know *a priori* what should be the most convenient one we have to choose for the second step. There are systems where this choice can be done in a special way such that the iterative procedure stops and it is not necessary to go to higher orders [9]. There are other systems, like the massive Yang-Mills theory, where it is feasible to carry out all the steps of the method [10,11].

It is important to emphasize that the arbitrariness in choosing some specific solution is not a blemish of the method. It just tell us that there can be more than one way of embedding some specific theory. On the other hand, it is also opportune to say that there are theories, like the full nonlinear sigma model, where no embedding is possible [12].

Recently, there have been a great deal of interest in noncommutative field theories. This has started when it was noted that noncommutative spaces naturally arise in the study of perturbative string theory in the presence of D-branes with a constant background magnetic field. The dynamics of the D-brane in this limit can be described by a noncommutative gauge theory [13]. Besides their origin in strings and branes, noncommutative field theories are a very interesting subject by their own rights. They have been studied extensively in many branches of field theory. We mention, among others, C, P, and T invariance [14], axial anomaly [15], noncommutative QED [16], supersymmetry [17], renormalization and the mixing of infrared and ultraviolet divergencies [18], unitarity [19], and phenomenology [20].

To obtain the noncommutative version of a field theory one essentially replaces the product of fields in the action by the Moyal product:

$$\phi_1(x) \star \phi_2(x) = \exp\left(\frac{i}{2}\theta^{\mu\nu}\partial^x_\mu\partial^y_\nu\right)\phi_1(x)\phi_2(y)|_{x=y} \quad (1.1)$$

where $\theta^{\mu\nu}$ is a real and antisymmetric constant matrix. It is easily verified that the Moyal product of two fields in the action is the same as the usual product, provide we discard boundary terms. In this way, the noncommutativity affects just the vertices.

In the present paper, we are going to consider the Hamiltonian embedding of the noncommutative version of the massive QED. By virtue of the Moyal commutators, this theory is non-Abelian and has an interesting

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mathematical structure that makes it completely different from the usual non-Abelian Yang-Mills case. The use of the BFFT method leads to infinite iterative steps. We are going to see that the solution of the equations corresponding to the initial steps as well as the inferring of the general solution are only achieved by using several identities related to the specific algebra of the Moyal space. We show that the second-class constraints and the Hamiltonian are transformed to form an involutive system of dynamical quantities that are defined as series of Moyal commutators among the variables belonging to the BFFT extended phase space. We also make explicit the gauge invariance of the first order system so obtained.

Our paper is organized as follows. In Sec. II we make a brief introduction of the massive noncommutative $U(1)$ theory in order to fix the notation and convention we are going to use through the paper. The embedding of the constraints is done in Sec. III and the obtainment of the involutive Hamiltonian, as well as the question of gauge invariance are considered in Sec. IV. We left Sec. V for some concluding remarks. We include four appendices to list the most important identities and present some details of relevant calculations.

II. THE NONCOMMUTATIVE MASSIVE U(1) THEORY

Let us start from the massless case. The corresponding action reads

$$S = -\frac{1}{4} \int d^4x \, F_{\mu\nu} \star F^{\mu\nu}$$
$$= -\frac{1}{4} \int d^4x \, F_{\mu\nu} F^{\mu\nu} \qquad (2.1)$$

However, the stress tensor is defined in terms of the Moyal commutator of the covariant derivatives, i.e.,

$$ie \, F_{\mu\nu} = [D_\mu, D_\nu]$$
$$= D_\mu \star D_\nu - D_\nu \star D_\mu \qquad (2.2)$$

Considering the covariant derivative

$$D_\mu = \partial_\mu - ie \, A_\mu \qquad (2.3)$$

one obtains

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ie \, (A_\mu \star A_\nu - A_\nu \star A_\mu)$$
$$= \partial_\mu A_\nu - \partial_\nu A_\mu - ie \, [A_\mu, A_\nu] \qquad (2.4)$$

We observe that the noncommutative stress tensor has the same general form as the corresponding Yang-Mills theory. There, gauge fields couple to themselves by means of color charges related to generators of the $SU(N)$ symmetry group. Here, the gauge field coupling is due to the mathematical definition of the Moyal product and does not have the same physical meaning of the

color charge coupling. The non-Abelian character of the two cases are completely different, both in physical and in mathematical points of view.

The gauge transformation of A_μ can be obtained by considering that the gauge transformation of the covariant derivative acting on some charged field ψ must satisfy the relation

$$(D_\mu \star \psi)' = U \star D_\mu \star \psi \qquad (2.5)$$

where $U(\alpha)$ is related to the noncommutative $U(1)$ symmetry group, namely,

$$U(\alpha) = \exp \star (i\alpha) \qquad (2.6)$$

Consequently,

$$U^{-1}(\alpha) = \exp \star (-i\alpha) \qquad (2.7)$$

Actually, by using the definition of the $\star$-product given by (1.1), we verify the relation

$$U \star U^{-1} = 1 \qquad (2.8)$$

From (2.5) and (2.8) one obtains the gauge transformation for A_μ

$$A'_\mu = \frac{i}{e} U \star D_\mu \star U^{-1} \qquad (2.9)$$

and it is not difficult to verify that indeed (2.1) is invariant under (2.9).

Let us now consider the massive case. The action reads

$$S = \int d^4x \left(-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} m^2 A_\mu A^\mu \right) \qquad (2.10)$$

By virtue of the mass term, this theory is not gauge invariant. We know that the gauge invariance can be attained by means of Stuckelberg compensating fields [21], whose general idea consists in extending the configuration space by the introduction of independent noncommutative $U(1)$ group elements g in the same representation of $U(\alpha)$ introduced above. The action (2.10) is then rewritten in terms of modified gauge fields $\bar{A}_\mu$, namely

$$\bar{A}_\mu = g \star A_\mu \star g^{-1} + ig \star \partial_\mu g^{-1} \qquad (2.11)$$

Since $\bar{F}_{\mu\nu} = g \star F_{\mu\nu} \star g^{-1}$, the kinetical term in (2.10) is not modified, due to the properties of the Moyal product. It is easy to proof, however, that the modified mass term becomes invariant under the infinitesimal gauge transformations

$$\delta A_\mu = D_\mu \star \alpha$$
$$= \partial_\mu \alpha - ie[A_\mu, \alpha]$$
$$\delta g = -ieg \star \alpha \qquad (2.12)$$

which lead to $\delta \bar{A}_\mu = 0$. Of course the old theory is recovered in the unitary gauge $g = 1$.

III. EMBEDDING OF MASSIVE NONCOMMUTATIVE U(1) THEORY

According to (2.10), the Lagrangian density corresponding to the massive noncommutative $U(1)$ theory is given by

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} m^2 A_\mu A^\mu \qquad (3.1)$$

The Moyal product does not explicitly appear in the equation above, but it is implicit into the expression of $F_{\mu\nu}$ given by (2.4). This means that we have to be careful in calculating canonical momenta starting from the Lagrangian density (3.1), because it contains an infinity number of time derivatives. Even though one can calculate momenta in theories with an infinite number of time derivatives [22], this would lead to causality and unitarity problems. These can be circumvented in the Moyal space by taking $\theta^{0i} = 0$ [19]. Hence, the $\star$-product of the gauge fields into the stress tensor $F_{\mu\nu}$ turns to be

$$A_\mu(x) \star A_\nu(x) = \exp\left(\frac{i}{2} \theta^{ij} \partial_i^x \partial_j^y\right) A_\mu(x) A_\nu(y)|_{x=y} \qquad (3.2)$$

Now, one can calculate the canonical momentum conjugate to A_μ in a direct way,

$$\pi^\mu = \frac{\partial \mathcal{L}}{\partial \dot{A}_\mu} = F^{\mu 0} \qquad (3.3)$$

We are using the following convention for the flat metric $\eta^{\mu\nu} = \text{diag.}\,(1, -1, -1, -1)$. We observe that π^0 is a primary constraint [1], that we denote by T_1,

$$T_1 = \pi^0 \qquad (3.4)$$

In order to look for secondary constraints, we construct the canonical Hamiltonian

$$H_c = \int d^3x \left(\pi^\mu \star \dot{A}_\mu - \mathcal{L}\right)$$
$$= \int d^3x \left(\pi^\mu \dot{A}_\mu - \mathcal{L}\right) \qquad (3.5)$$

where the last step was only possible inside an integration over d^3x because we are taking $\theta^{0i} = 0$. Using the expression (3.1) we obtain for the total Hamiltonian

$$H_T = H_c + \int d^3x \, \lambda T_1$$
$$= \int d^3x \left(-\frac{1}{2} \pi^i \pi_i - A_0 \, \partial_i \pi^i + ie \, \pi^i [A_0, A_i]\right.$$
$$\left. + \frac{1}{4} F^{ij} F_{ij} - \frac{1}{2} m^2 A^\mu A_\mu + \lambda T_1\right) \qquad (3.6)$$

The term $\pi^0 \dot{A}_0$ of the canonical Hamiltonian density was absorbed into λT_1 by a redefinition of the Lagrange multiplier λ.

To calculate the consistency condition for the constraint $\pi^0 = 0$, we have to make the Poisson brackets between π^0 and H_T equal to zero. Since Poisson brackets are taken at equal time and we are considering $\theta^{0i} = 0$, the Poisson bracket definition between two generic quantities F and G is given in terms of the usual product, (but we mention that the Dirac brackets are not) i.e.

$$\{F(x), G(y)\}_{x_0 = y_0}$$
$$= \int d^3z \left(\frac{\delta F(\vec{x})}{\delta A_\mu(\vec{z})} \frac{\delta G(\vec{y})}{\delta \pi^\mu(\vec{z})} - \frac{\delta G(\vec{y})}{\delta A_\mu(\vec{z})} \frac{\delta F(\vec{x})}{\delta \pi^\mu(\vec{z})}\right) \qquad (3.7)$$

The consistency condition for T_1 is

$$\{\pi_0(x), H_T\} = -\frac{\delta}{\delta A_0(x)} H_T$$

$$= \partial_i \pi^i(x) + m^2 A_0(x) - ie \int d^3y \, \pi^i(y) \Big(\delta^3(x-y) \star A_i(y) - A_i(y) \star \delta^3(x-y)\Big)$$

$$= \partial_i \pi^i(x) + m^2 A_0(x) - ie \int d^3y \, \pi^i(y) \star \delta^3(x-y) \star A_i(y) + ie \int d^3y \, \pi^i(y) \star A_i(y) \star \delta^3(x-y)$$

$$= \partial_i \pi^i(x) + m^2 A_0(x) - ie \int d^3y \, A_i(y) \star \pi^i(y) \star \delta^3(x-y) + ie \int d^3y \, \pi^i(y) \star A_i(y) \star \delta^3(x-y)$$

$$= \partial_i \pi^i(x) + m^2 A_0(x) - ie \int d^3y \, [A_i(y), \pi^i(y)] \, \delta^3(x-y) + m^2 A_0(x)$$

$$= \partial_i \pi^i(x) + m^2 A_0(x) - ie \, [A_i(x), \pi^i(x)]$$
$$= D_i \star \pi^i(x) + m^2 A_0(x) \qquad (3.8)$$

where it is understood the equal time condition for the Poisson brackets. In the last steps it was used the identities (A.1)-(A.3).

The result given by (3.8) is the constraint $T_2(x)$ and there are no more constraints. This is verified by showing that constraints T_1 and T_2 satisfy a non-involutive algebra. In fact

$$\{T_1(x), T_1(y)\} = 0 \qquad (3.9)$$

$$\{T_1(x), T_2(y)\} = -m^2\delta^3(x-y) \qquad (3.10)$$
$$\{T_2(x), T_2(y)\} = ie\,[D_i \star \pi^i(x), \delta^3(x-y)] \qquad (3.11)$$

The Eqs. (3.9) and (3.10) are directly obtained without effort. See Appendix B for the obtainment of (3.11).

Let us now start to use the BFFT method. From expressions (3.9)-(3.11) we identify the quantities (we are going to use the same notation as the one of reference [10])

$$\Delta_{11}(x,y) = 0$$
$$\Delta_{12}(x,y) = -m^2\delta^3(x-y) = -\Delta_{21}(x,y)$$
$$\Delta_{22}(x,y) = ie\,[D_i \star \pi^i(x), \delta^3(x-y)] \qquad (3.12)$$

We extend the phase space by introducing one new auxiliary field for each (second-class) constraint. We take them as conjugate (unconstrained) fields, i.e.

$$\{\eta^1(x), \eta^2(y)\} = \delta^3(x-y)$$
$$\{\eta^1(x), \eta^1(y)\} = 0 = \{\eta^2(x), \eta^2(y)\} \qquad (3.13)$$

which permit us to identify the symplectic matrix $(\omega^{ab}) = (\{\eta^a, \eta^b\})$

$$\left(\omega^{ab}(x,y)\right) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \delta^3(x-y) \qquad (3.14)$$

The first correction of the constraints,

$$T_a^{(1)}(x) = \int d^3y\, X_{ab}(x,y) \star \eta^b(y) \qquad (3.15)$$

is achieved by solving the equation

$$\Delta_{ab}(x,y) + \int d^3z\, d^3z'\, X_{ac}(x,z) \star \omega^{cd}(z,z') \star X_{bd}(y,z') = 0$$
$$\implies \Delta_{ab}(x,y) + \int d^3z\, d^3z'\, X_{ac}(x,z)\, \omega^{cd}(z,z')\, X_{bd}(y,z') = 0 \qquad (3.16)$$

Considering (3.12), (3.14), and (3.16), we have

$$a=1,\, b=1: \quad \int d^3z\, \Big(X_{11}(x,z)X_{12}(y,z) - X_{12}(x,z)X_{11}(y,z)\Big) = 0 \qquad (3.17)$$

$$a=1,\, b=2: \quad \int d^3z\, \Big(X_{11}(x,z)X_{22}(y,z) - X_{12}(x,z)X_{21}(y,z)\Big) = m^2\,\delta^3(x-y) \qquad (3.18)$$

$$a=2,\, b=2: \quad \int d^3z\, \Big(X_{21}(x,z)X_{22}(y,z) - X_{22}(x,z)X_{21}(y,z)\Big) = -ie\,[D_i \star \pi^i(x), \delta^3(x-y)] \qquad (3.19)$$

The set given by (3.17) - (3.19) has more unknowns than the number of equations. Let us choose

$$X_{11}(x,y) = 0$$
$$X_{21}(x,y) = \delta^3(x-y) \qquad (3.20)$$

We thus see that (3.17) is automatically verified. Eqs. (3.18) and (3.19) lead to

$$X_{12}(x,y) = -m^2\,\delta^3(x-y) \qquad (3.21)$$

$$X_{22}(x,y) - X_{22}(y,x) = ie\,[D_i \star \pi^i(x), \delta^3(x-y)] \quad (3.22)$$

Looking at both sides of Eq. (3.22) we are tempted to identify $X_{22}(x,y)$ with $ie\, D_i \star \pi^i(x) \star \delta^3(x-y)$. However, we also observe that only the antisymmetric part of $X_{22}(x,y)$ contributes. Keeping just this part we have

$$X_{22}(x,y)$$
$$= \frac{ie}{2}\Big(D_i \star \pi^i(x) \star \delta^3(x-y) - D_i \star \pi^i(y) \star \delta^3(x-y)\Big)$$
$$= \frac{ie}{2}\Big(D_i \star \pi^i(x) \star \delta^3(x-y) - \delta^3(x-y) \star D_i \star \pi^i(x)\Big)$$
$$= \frac{ie}{2}\,[D_i \star \pi^i(x), \delta^3(x-y)] \qquad (3.23)$$

In the first to the second step it was used the identity (A.4). The combination of (3.15), (3.20), (3.21), and (3.23) leads to the first correction of the constraints

$$T_1^{(1)}(x) = -m^2\eta^2(x) \qquad (3.24)$$
$$T_2^{(1)}(x) = \eta^1(x) - \frac{ie}{2}[\eta^2(x), D_i \star \pi^i(x)] \qquad (3.25)$$

See Appendix C for the obtainment of $T_2^{(1)}(x)$

The next task is to evaluate the second correction $T_a^{(2)}$. This can be achieved from the general expression

$$\{T_a, T_b^{(1)}\} + \{T_a^{(1)}, T_b\} + \{T_a^{(1)}, T_b^{(2)}\}_{(\eta)}$$
$$+ \{T_a^{(2)}, T_b^{(1)}\}_{(\eta)} = 0 \qquad (3.26)$$

Using the expressions for the constraints T_a, given by (3.4) and (3.8), as well as (3.24) and (3.25), we obtain

$$a=1,\, b=1:$$
$$\{\eta^2, T_1^{(2)}\}_{(\eta)} + \{T_1^{(2)}, \eta^2\}_{(\eta)} = 0$$
$$\qquad (3.27)$$
$$a=1,\, b=2:$$

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$$m^2\{\eta^2, T_2^{(2)}\}_{(\eta)} - \{T_1^{(2)}, \eta^1 - \frac{ie}{2}[\eta^2, D_i \star \pi^i]\}_{(\eta)} = 0 \tag{3.28}$$

$a = 2, b = 1$:
$$\{\eta^1 - \frac{ie}{2}[\eta^2, D_i \star \pi^i], T_1^{(2)}\}_{(\eta)} - m^2\{T_2^{(2)}, \eta^2\}_{(\eta)} = 0 \tag{3.29}$$

$a = 2, b = 2$:
$$\frac{ie}{2}\{D_i \star \pi^i, [\eta^2, D_j \star \pi^j]\} + \frac{ie}{2}\{[\eta^2, D_j \star \pi^j], D_i \star \pi^i\}$$
$$- \{\eta^1 - \frac{ie}{2}[\eta^2, D_i \star \pi^i], T_2^{(2)}\}_{(\eta)}$$
$$- \{T^{(2)}, \eta^1 - \frac{ie}{2}[\eta^2, D_i \star \pi^i]\}_{(\eta)} = 0 \tag{3.30}$$

The solution of (3.30) is (see Appendix D)

$$T_2^{(2)}(x) = \frac{(ie)^2}{6}[\eta^2(x), [\eta^2(x), D_i \star \pi^i(x)] \tag{3.31}$$

Looking at the remaining equations (3.27)-(3.29) we observe that the solution we have obtained for $T_2^{(2)}$ does not give any contribution to those equations. So, if one takes $T_1^{(2)} = 0$ into those remaining equations, they will be automatically verified.

We can go on and calculate other corrections for the constrains. The procedure is similar to this last calculation of $T_a^{(2)}$. We just write down that the general solution for the embedding procedure of the massive noncommutative U(1) theory is given by

$$\tilde{T}_1 = \pi_0 - m^2\eta^2 \tag{3.32}$$
$$\tilde{T}_2 = D_i \star \pi^i + m^2 A_0 + \eta^1 - \frac{ie}{2!}[\eta^2, D_i \star \pi^i]$$
$$+ \frac{(-ie)^2}{3!}[\eta^2, [\eta^2, D_i \star \pi^i]]$$
$$+ \frac{(-ie)^3}{4!}[\eta^2, [\eta^2, [\eta^2, D_i \star \pi^i]]]$$
$$+ \dots \tag{3.33}$$

IV. TIME EVOLUTION AND GAUGE INVARIANCE

So far we have succeeded in finding the converted constraints $\tilde{T}_1$ and $\tilde{T}_2$, which are in involution, as can be verified. To get a complete description of the dynamics, it is also necessary to obtain an involutive Hamiltonian. This can be done by using directly the BFFT procedure. A simpler approach, however, is the one used in Ref. [10] where modified phase space variables $\tilde{A}_\mu$ and $\tilde{\pi}_\mu$ are constructed in such a way that they are involutive with the converted constraints $\tilde{T}_1$ and $\tilde{T}_2$ and at the same time

recover the original phase space variables A_μ and Π_μ in the unitary gauge. With the aid of $\tilde{A}_\mu$ and $\tilde{\pi}_\mu$, we then directly obtain the proper involutive Hamiltonian by the rule $\tilde{H}(A_\mu, \pi_\mu, \eta^a) = H(\tilde{A}_\mu, \tilde{\pi}_\mu)$, where $H(A_\mu, \pi_\mu)$ given by (3.6). As the calculations are long and similar to those performed in the previous sections, we just display the resulting expressions for $\tilde{A}_\mu$ and $\tilde{\pi}_\mu$:

$$\tilde{A}_0 = A_0 + \frac{1}{m^2}\eta^1 + \frac{ie}{2m^2}[\eta^2, D_i \star \pi^i]$$
$$- \frac{(ie)^2}{3m^2}[\eta^2, [\eta^2, D_i \star \pi^i]] + \cdots$$
$$- \frac{n}{(n+1)!}\frac{(-ie)^n}{m^2}[\eta^2, [\eta^2, \dots, [\eta^2, D_i \star \pi^i]] \dots] \tag{4.1}$$

$$\tilde{A}_i = A_i - D_i \star \eta^2 + \frac{ie}{2}[\eta^2, D_i \star \eta^2]$$
$$- \frac{(-ie)^2}{3!}[\eta^2, [\eta^2, D_i \star \eta^2]] + \cdots \tag{4.2}$$

$$\tilde{\pi}_0 = \pi_0 \tag{4.3}$$

$$\tilde{\pi}_i = \pi_i - ie[\eta^2, \pi_i] + \frac{(-ie)^2}{2}[\eta^2, [\eta^2, \pi_i]]$$
$$+ \frac{(-ie)^3}{3!}[\eta^2, [\eta^2, [\eta^2, \pi_i]]] + \cdots \tag{4.4}$$

Since $\tilde{T}_1$ and $\tilde{T}_2$ are involutive with respect to $\tilde{H}$, they are consistently conserved. Also, as they are first class, they generate the gauge transformations inside the Hamiltonian formalism. Let ω^a, $a = 1, 2$ be arbitrary infinitesimal functions. Then $G[\omega^a] = \int d^3x\,(\omega^a\tilde{T}_a)$ generates infinitesimal gauge transformations in the phase space variables y^α given by $\delta y^\alpha = \{y^\alpha, G\}$. It is then possible to show [2] that the first order action

$$S_{f0} = \int d^4x\,(\pi^\mu \dot{A}_\mu + \eta^2\dot{\eta}^1 - \tilde{H} - \lambda^a\tilde{T}_a) \tag{4.5}$$

is gauge invariant since the Hamiltonian and the constraints satisfy an Abelian algebra, once $\delta\lambda^a = \dot{\omega}^a$. It is not difficult to verify that the gauge transformations of the phase space coordinates are given by

$$\delta A_0 = -\omega_1$$
$$\delta A_i = -D_i \star \left(\omega_2 - \frac{ie}{2!}[\omega_2, \eta^2] + \frac{(-ie)^2}{3!}[[\omega^2, \eta^2], \eta^2]\right.$$
$$\left. + \frac{(-ie)^3}{4!}[[[\omega^2, \eta^2], \eta^2], \eta^2] + \cdots\right)$$
$$\delta\pi_0 = -m^2\omega_2$$
$$\delta\pi_i = ie[\pi_i, \omega_2] - \frac{(ie)^2}{2!}[\eta^2, [\pi_i, \omega_2]]$$
$$+ \frac{(ie)^3}{3!}[\eta^2, [\eta^2, [\pi_i, \omega_2]]] + \cdots$$
$$\delta\eta^1 = -m^2\omega_1 - \frac{ie}{2!}[D_i \star \pi^i, \omega_2]$$
$$- \frac{(ie)^2}{3!}[D_i \star \pi^i, [\eta^2, \omega_2]]$$

$$-\frac{(ie)^3}{4!}\left[D_i \star \pi^i, [\eta^2, [\eta^2, \omega_2]]\right] - \cdots$$

$$\delta\eta^2 = \omega_2 \qquad (4.6)$$

Observe that these transformations have the proper commutative limit but are not trivially related to the Lagrangian ones, given by (2.12).

V. CONCLUSION

In this paper we have studied the embedding of the massive noncommutative $U(1)$ theory by using the BFFT Hamiltonian formalism. Due to the peculiar mathematical structure of the algebra related to the Moyal space, some care was necessary in order to deal with the iterative steps of the method. For this part we emphasize the importance of the identities (A.4) and (A.8). We succeeded in finding the converted constraints and Hamiltonian, which are in involution. They were obtained in terms of infinite series of Moyal commutators among the BFFT variables and functions of the old phase space. The gauge structure of the embedded theory was also displayed.

APPENDIX A

SOME IDENTITIES RELATED TO THE STAR PRODUCT

In this Appendix we list the main identities used in the paper

$$\int d^4x\, \phi_1 \star \phi_2 = \int d^4x\, \phi_1\phi_2 = \int d^4x\, \phi_2 \star \phi_1 \qquad (A.1)$$

$$\left(\phi_1 \star \phi_2\right) \star \phi_3 = \phi_1 \star \left(\phi_2 \star \phi_3\right) = \phi_1 \star \phi_2 \star \phi_3 \qquad (A.2)$$

$$\int d^4x\, \phi_1 \star \phi_2 \star \phi_3 = \int d^4x\, \phi_2 \star \phi_3 \star \phi_1$$
$$= \int d^4x\, \phi_3 \star \phi_1 \star \phi_2 \qquad (A.3)$$

$$\phi(x) \star \delta(x-y) = \delta(x-y) \star \phi(y) \qquad (A.4)$$

$$D_\mu \star (\phi_1 \star \phi_2) = (D_\mu \star \phi_1) \star \phi_2 + \phi_1 \star (D_\mu \star \phi_2) \qquad (A.5)$$

$$D_\mu^x \star \delta(x-y) = -D_\mu^y \star \delta(x-y) \qquad (A.6)$$

$$[\phi_1, [\phi_2, \phi_3]] + [\phi_2, [\phi_3, \phi_1]] + [\phi_3, [\phi_1, \phi_2]] = 0 \qquad (A.7)$$

$$[\phi_1(x), [\phi_2(x), \delta(x-y)]] = [\phi_2(y), [\phi_1(y), \delta(x-y)]] \qquad (A.8)$$

APPENDIX B

OBTAINMENT OF (3.11)

$$\{T_2(x), T_2(y)\} = \{D_i \star \pi^i(x), D_j \star \pi^j(y)\}$$
$$= \int d^3z \left(\frac{\delta^3}{\delta^3 A_k(z)}\left(D_i \star \pi^i(x)\right)\frac{\delta^3}{\delta^3 \pi^k(z)}\left(D_j \star \pi^j(y)\right) - \frac{\delta^3}{\delta^3 A_k(z)}\left(D_j \star \pi^j(y)\right)\frac{\delta^3}{\delta^3 \pi^k(z)}\left(D_i \star \pi^i(x)\right) \right)$$
$$= -ie \int d^3z \left([\delta^3(x-z), \pi^k(x)]\, D_k^y \star \delta^3(y-z) - [\delta^3(y-z), \pi^k(y)]\, D_k^x \star \delta^3(x-z) \right)$$
$$= -ie\, D_k^y \star \int d^3z\, [\delta^3(x-z), \pi^k(x)]\, \delta^3(y-z) + ie\, D_k^x \star \int d^3z\, [\delta^3(y-z), \pi^k(y)]\, \delta^3(x-z)$$
$$= -ie\, D_k^y \star \left([\delta^3(x-y), \pi^k(x)] \right) + ie\, D_k^x \star \left([\delta^3(y-x), \pi^k(y)] \right)$$
$$= -ie\, [D_k^y \star \delta^3(x-y), \pi^k(x)] + ie\, D_k^x \star \left([\pi^k(x), \delta^3(x-y)] \right) \qquad (B.1)$$

where it was used the identity (A.4) for the second term of the last step above. Considering (A.5) and (A.6) we have

$$\{T_2(x), T_2(y)\} = ie\, [D_k^x \star \delta^3(x-y), \pi^k(x)] + ie\, [D_k \star \pi^k(x), \delta^3(x-y)] + ie\, [\pi^k(x), D_k^x \star \delta^3(x-y)]$$
$$= ie\, [D_k \star \pi^k(x), \delta^3(x-y)] \qquad (B.2)$$

Making the replacement of (3.20) and (3.23) into (3.15) we get

$$T_2^{(1)} = \int d^3y\, \delta^3(x-y) \star \eta^1(y)$$
$$+ \frac{ie}{2} \int d^3y\, [D_i \star \pi(x), \delta^3(x-y)] \star \eta^2(y)$$
$$= \eta^1(x) + \frac{ie}{2} \int d^3y\, D_i \star \pi^i(x) \star \delta^3(x-y) \star \eta^2(y)$$
$$- \frac{ie}{2} \int d^3y\, \delta^3(x-y) \star D_i \star \pi^i(x) \star \eta^2(y) \quad \text{(C.1)}$$

In the triple Moyal product above involving $D_i \star \pi^i(x)$, $\eta^2(y)$ and $\delta^3(x-y)$ one cannot use the rule given by (A.3) because these products do not refer to the same variable. Let us then conveniently developed each integral in a separate way.

$$\int d^3y\, D_i \star \pi^i(x) \star \delta^3(x-y) \star \eta^2(y)$$
$$= D_i \star \pi^i(x) \star \int d^3y\, \delta^3(x-y) \star \eta^2(y)$$
$$= D_i \star \pi^i(x) \star \int d^3y\, \delta^3(x-y)\, \eta^2(y)$$
$$= D_i \star \pi^i(x) \star \eta^2(x) \quad \text{(C.2)}$$

For the second integral, we have

$$\int d^3y\, \delta^3(x-y) \star D_i \star \pi^i(x) \star \eta^2(y)$$
$$= \int d^3y\, \delta^3(x-y) \star D_i \star \pi^i(x)\, \eta^2(y)$$
$$= \left(\int d^3y\, \eta^2(y) \delta^3(x-y) \right) \star D_i \star \pi^i(x)$$
$$= \eta^2(x) \star D_i \star \pi^i(x) \quad \text{(C.3)}$$

The replacement of (C.2) and (C.3) into (C.1) leads to $T_2^{(1)}(x)$ given by (3.25).

APPENDIX D

SOLUTION OF EQ. (3.30)

Let us develop the two first terms of (3.30) in a separate way. For the first term, we have

$$-\frac{ie}{2}\{D_i \star \pi^i, [\eta^2, D_j \star \pi^j]\} = -\frac{ie}{2}\{D_i \star \pi^i(x), \eta^2(y) \star D_j \star \pi^j(y) - D_j \star \pi^j(y) \star \eta^2(y)\}$$
$$= -\frac{ie}{2}\eta^2(y) \star \{D_i \star \pi^i(x), D_j \star \pi^j(y)\} + \frac{ie}{2}\{D_i \star \pi^i(x), D_j \star \pi^j(y)\} \star \eta^2(y)$$
$$= \frac{ie}{2}[\{D_i \star \pi^i(x), D_j \star \pi^j(y)\}, \eta^2(y)] \quad \text{(D.1)}$$

Using (A.4), (A.8), and (3.11), we obtain

$$-\frac{ie}{2}\{D_i \star \pi^i, [\eta^2, D_j \star \pi^j]\} = \frac{(ie)^2}{2}[[D_i \star \pi^i(x), \delta^3(x-y)], \eta^2(y)]$$
$$= \frac{(ie)^2}{2}[[\delta^3(x-y), D_i \star \pi^i(y)], \eta^2(y)]$$
$$= \frac{(ie)^2}{2}[[\delta^3(x-y), \eta^2(x)], D_i \star \pi^i(x)] \quad \text{(D.2)}$$

Developing the second term in a similar way, the result is

$$-\frac{ie}{2}\{[\eta^2, D_j \star \pi^j], D_i \star \pi^i\} = \frac{(ie)^2}{2}[[D_i \star \pi^i(x), \delta^3(x-y)], \eta^2(x)] \quad \text{(D.3)}$$

Let us introduce these two terms into expression (3.30),

$$\{T_2^{(2)}, \eta^1 - \frac{ie}{2}[\eta^2, D_i \star \pi^i]\}_{(\eta)} + \{\eta^1 - \frac{ie}{2}[\eta^2, D_i \star \pi^i], T_2^{(2)}\}_{(\eta)}$$
$$= -\frac{(ie)^2}{2}\left([[\delta^3(x-y), \eta^2(x)], D_i \star \pi^i(x)] + [[D_i \star \pi^i(x), \delta^3(x-y)], \eta^2(x)] \right)$$
$$= \frac{(ie)^2}{2}[[\eta^2(x), D_i \star \pi^i(x)], \delta^3(x-y)] \quad \text{(D.4)}$$

where it was used the Jacobi identity (A.7).

Looking at both sides of (D.3) we observe that the only possibility of solution is considering $T_2^{(2)}$ quadratic in η^2 (it could not depend on $\eta^1\eta^2$ or $\eta^1\eta^1$ because this would lead to terms in η^1 in the left side what would be inconsistent with the right side). We infer that an appropriate form for $T_2^{(2)}$ should be

$$T_2^{(2)}(x) = \frac{(ie)^2}{6}\,[\eta^2(x),[\eta^2(x),D_i\star\pi^i(x)]] \tag{D.5}$$

where $(ie)^2/6$ is a convenient factor to achieve the solution of (3.30). Actually, developing the left side of (D.4) with $T_2^{(2)}$ given by (D.5), we get

$$\frac{(ie)^2}{6}\{[\eta^2(x),[\eta^2(x),D_i\star\pi^i(x)]],\eta^1(y)\} + \frac{(ie)^2}{6}\{\eta^1(x),[\eta^2(y),[\eta^2(y),D_i\star\pi^i(y)]]\}$$

$$= -\frac{(ie)^2}{6}\int d^3z\,\frac{\delta^3}{\delta^3\eta^2(z)}[\eta^2(x),[\eta^2(x),D_i\star\pi^i(x)]]\delta^3(y-z)$$

$$+ \frac{(ie)^2}{6}\int d^3z\,\delta^3(x-z)\frac{\delta^3}{\delta^3\eta^2(z)}[\eta^2(y),[\eta^2(y),D_i\star\pi^i(y)]]$$

$$= -\frac{(ie)^2}{6}[\delta^3(x-y),[\eta^2(x),D_i\star\pi^i(x)]] - \frac{(ie)^2}{6}[\eta^2(x),[\delta^3(x-y),D_i\star\pi^i(x)]]$$

$$+ \frac{(ie)^2}{6}[\delta^3(x-y),[\eta^2(y),D_i\star\pi^i(y)]] + \frac{(ie)^2}{6}[\eta^2(y),[\delta^3(x-y),D_i\star\pi^i(y)]] \tag{D.6}$$

From the identity (A.4) we have that the first and the third terms of (D.6) are equal. The last one can be written as $-\frac{(ie)^2}{6}[D_i\star\pi^i(x),[\eta^2(x),\delta(x-y)]]$, by virtue of (A.8). Using the Jacobi identity with this and the second term of (D.6) we obtain a term that is equal to the first one. Hence, the four terms of (D.6) is three times the first term, that is precisely the right side of (D.4) as we want to show.

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